Problem-Solving via Search

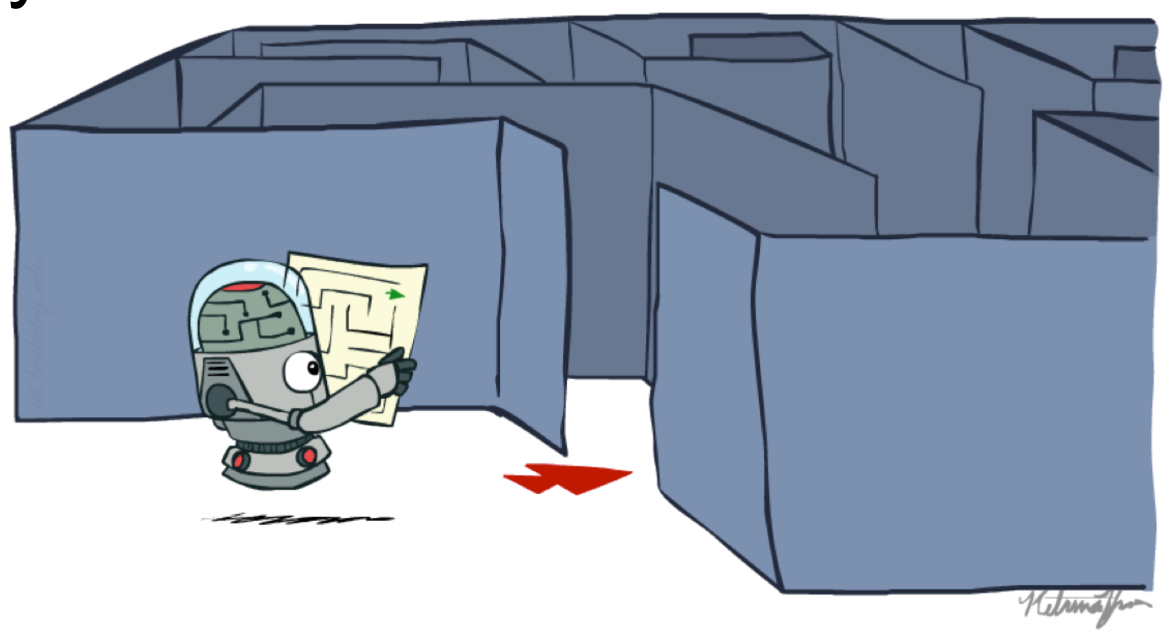
Lecture 3

What is a **search problem**?

How do **search algorithms** work and how do we **evaluate their performance**?

Agenda

- An example task
- Problem formulation
- Infrastructure for search algorithms
  - Complexity analysis



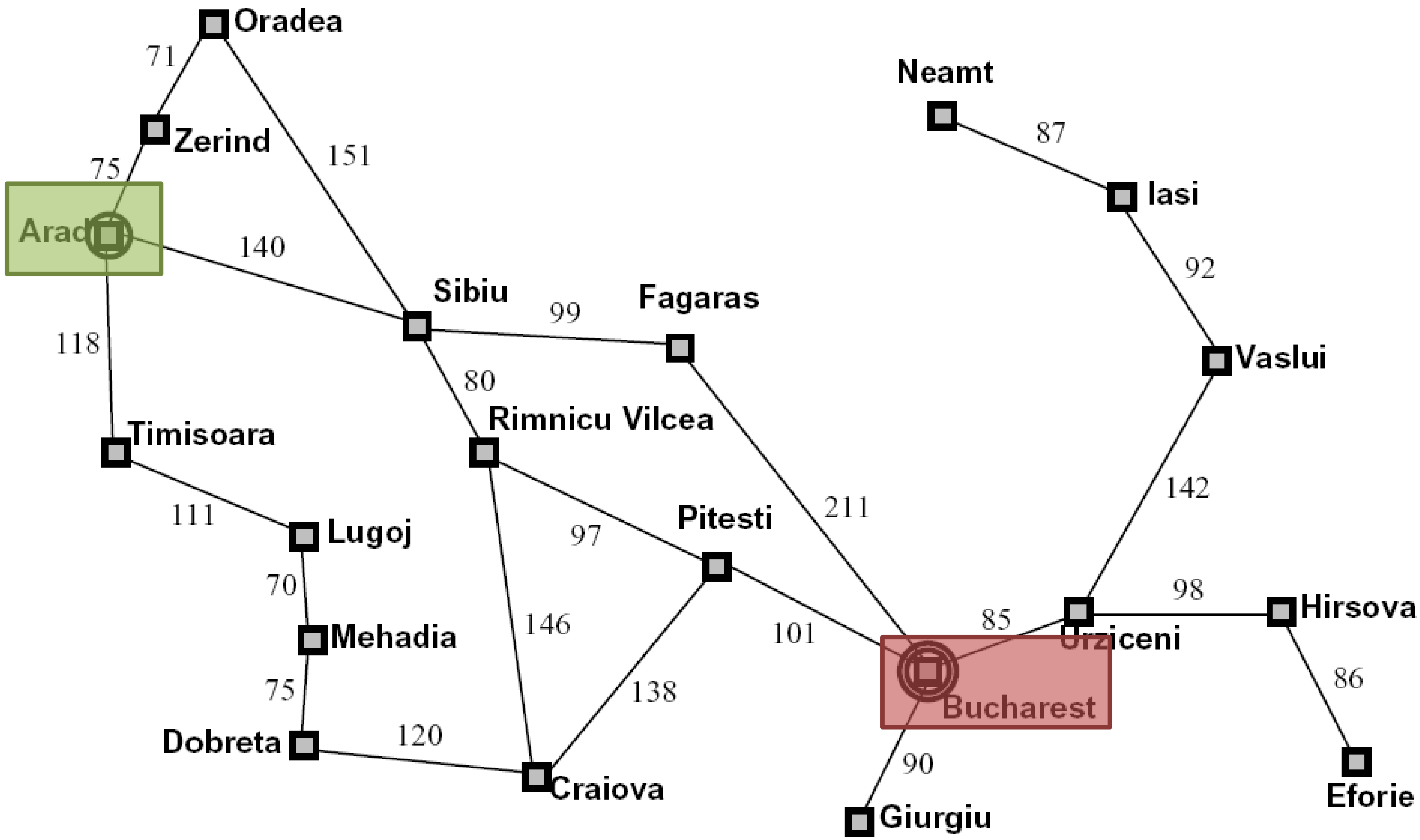
A Motivating Task

- Start: Arad, Romania
- Goal: Bucharest, Romania
  – Roads leading to Sibiu, Timisoara, Zerind

What is a rational agent to do?

Add Geographical Knowledge

Add Abstraction



Describe the Task

- Observability
- Certainty
- Representation
- A priori

- Full
- Deterministic
- Discrete
- Known

Under these conditions we can ***search*** *for a problem* ***solution****, a fixed sequence of actions*

- *Given a perfect model, can be done* ***open-loop*** *(i.e. ignore percepts)*

Search Problem Formalism

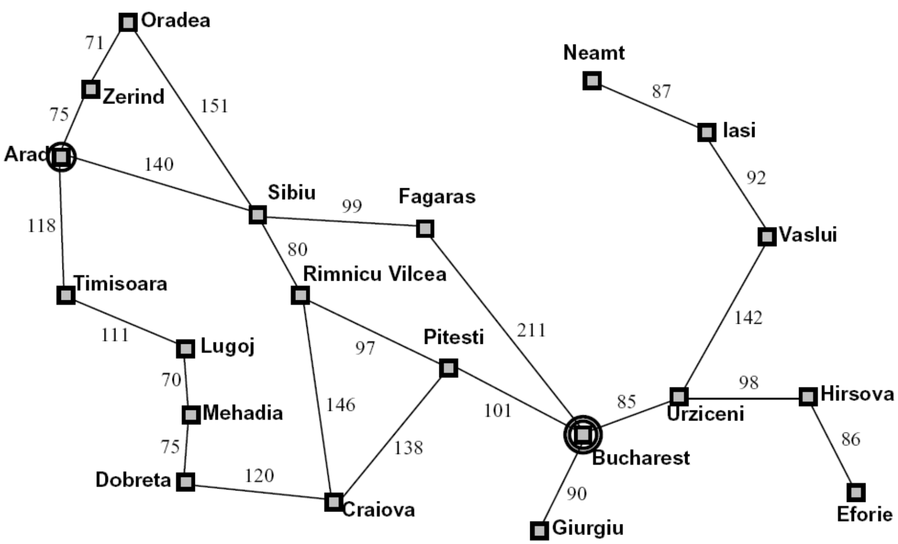
Defined via the following components:

- The **initial state** the agent starts in
- A **successor/transition function**
  $S(x)$ = {action -> state, cost}
- A **goal test**, which returns true if a given state is a goal state
  G(x) = true/false
- A **path cost** that assigns a numeric cost to each path
  - Typically assumed to be sum of action costs

A **solution** is a sequence of actions leading from initial state to a goal state (**optimal** = lowest path cost)

Together the initial state and successor function implicitly define the **state space**, the set of all reachable states

Example: Romanian Travel



- Initial state

  Arad

- Successor

  Adjacency, cost=distance

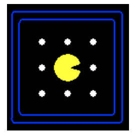
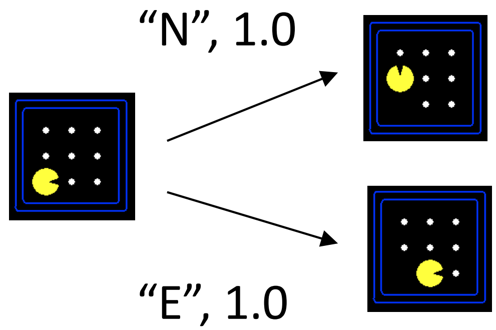
- Goal test

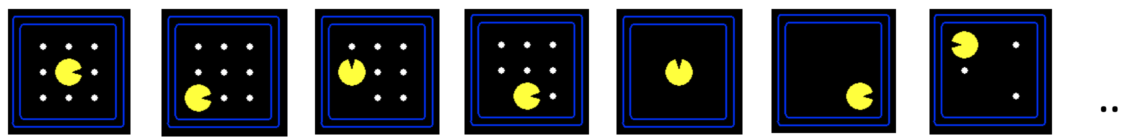
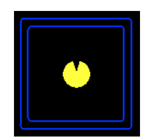
  City ?= Bucharest

- State space

  Cities

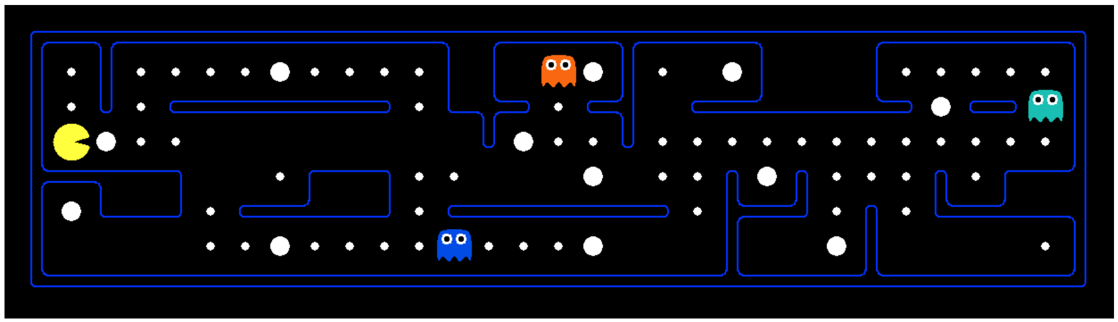
Example: Pacman

- Initial state 
- Successor function 

- State space 
- Goal test: no more food (e.g. )

State Abstraction

- Often world states are absurdly complex

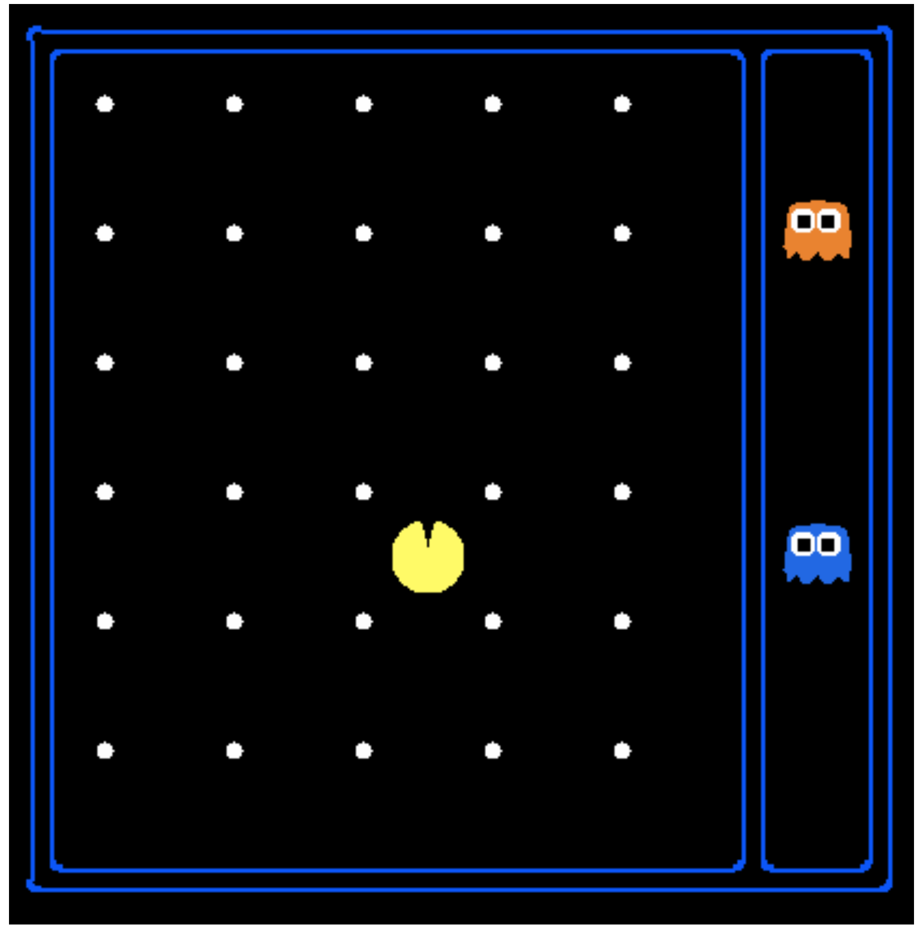


- To make a particular problem tractable, we **abstract** the state to only represent details necessary to solve the problem

Abstraction is Necessary

World state
- Agent positions: 120
- Food count: 30
- Ghost positions: 12
- Agent facing: NSEW

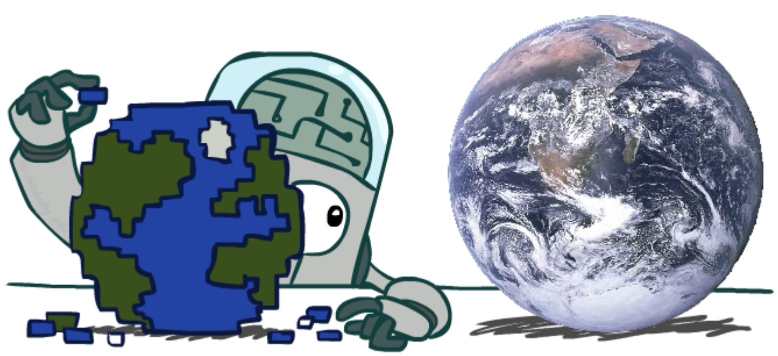
How many…
- world states?
  $120x(2^{30})x(12^2)x4$
- states for path planning?
  120
- states for eat-all-dots?
  $120x(2^{30})$



Exercise: Abstractions

Path Planning

- States: (x,y)
- Actions: NSEW
- Successor: (x',y')
- Goal test: (x,y)=END



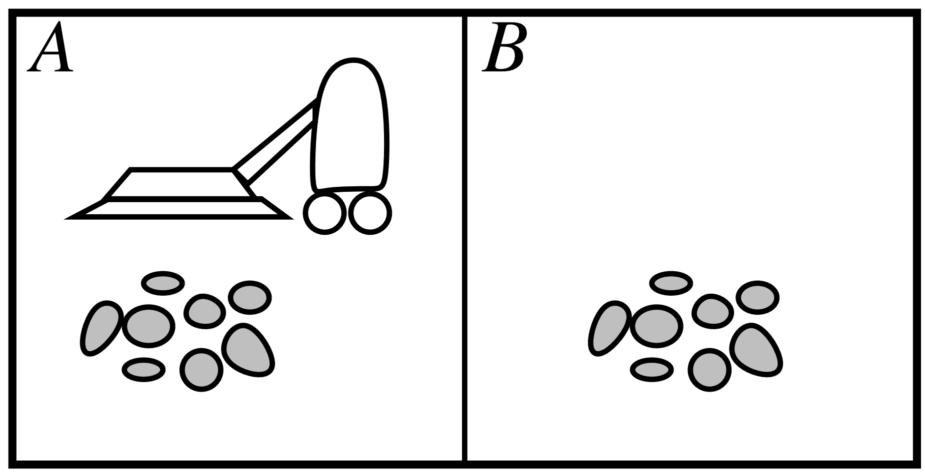
Eat All the Dots

- States: {(x,y), T/F grid}
- Actions: NSEW
- Successor: (x',y'), possibly T/F change
- Goal test: grid = all F's

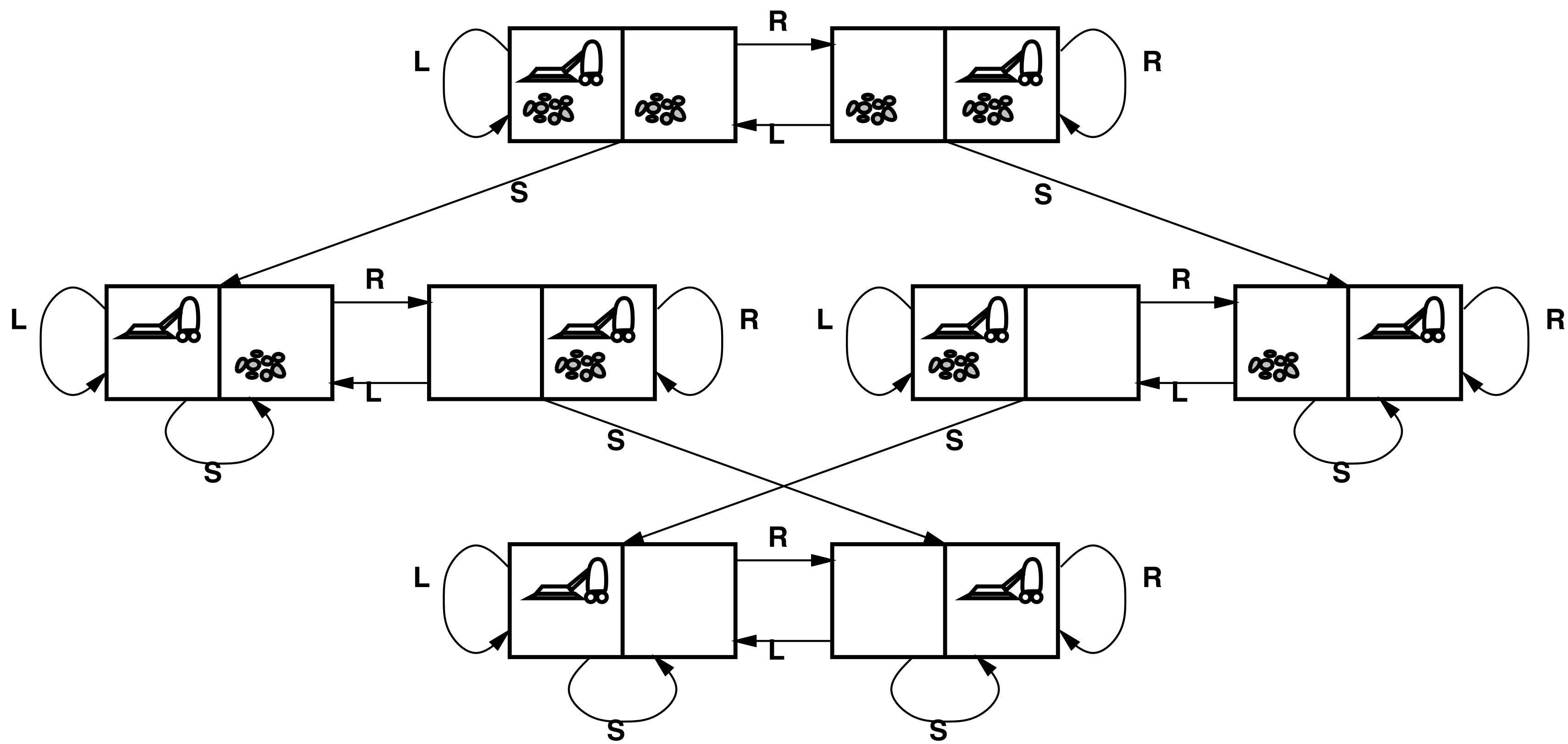
Exercise

Define the **`vacuum-cleaner`** search problem:

- World state representation
- Search state representation
- Transition model
  - State space
- Goal test

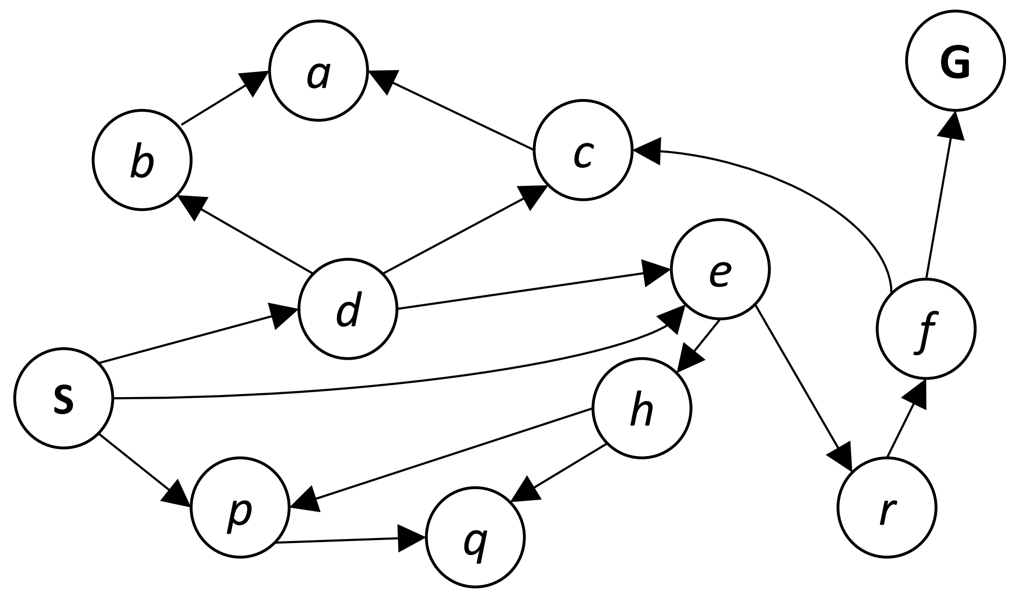


Solution State Space Graph



State Space Graph

- State space graph: A mathematical representation of a search problem
  - Nodes are (abstracted) world configurations
  - Arcs represent successors (action results)
  - The goal test is a set of goal node(s)

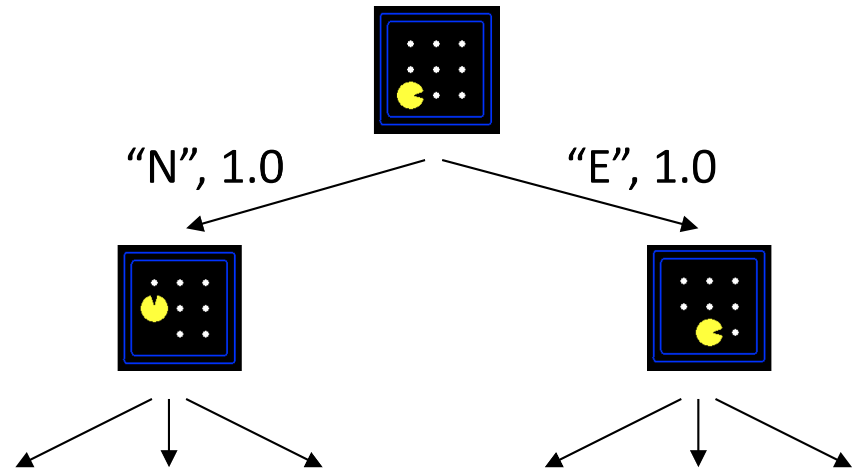


- In a search graph, each state occurs only once!

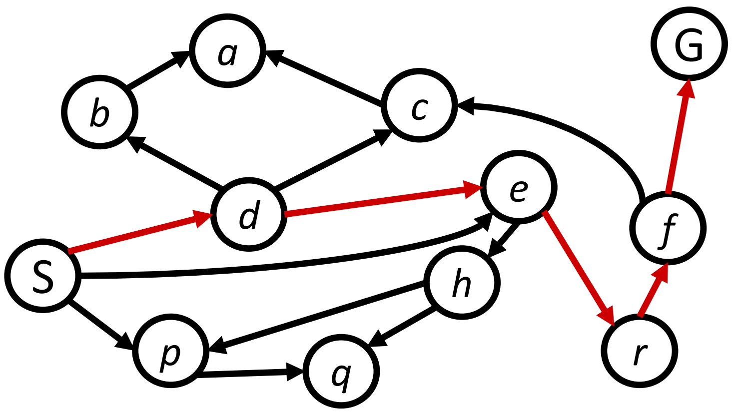
- **We can rarely build this full graph in memory (i.e. it's too big), but it's a useful idea**

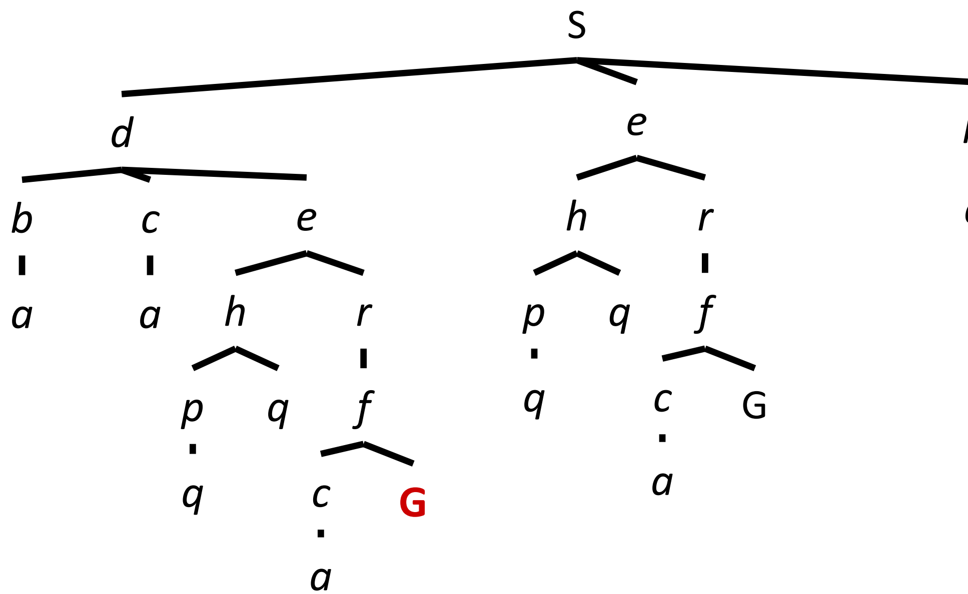
Search Tree

- A “what if” tree of plans and their outcomes
- The start state is the root node
- Children correspond to successors
- Nodes show states, but correspond to PLANS that achieve those states
- **For most problems, we can never actually build the whole tree**



State Space Graph vs. Search Tree

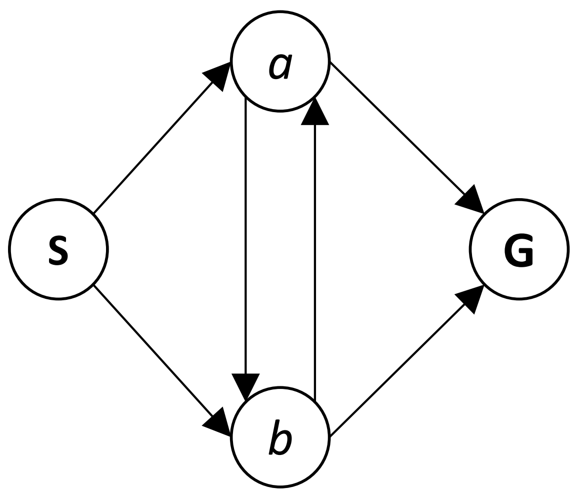




- Each NODE in in the search tree is an entire PATH in the state space graph.
- We construct both on demand – and we construct as little as possible.

Exercise

Consider the following 4-state state space graph…



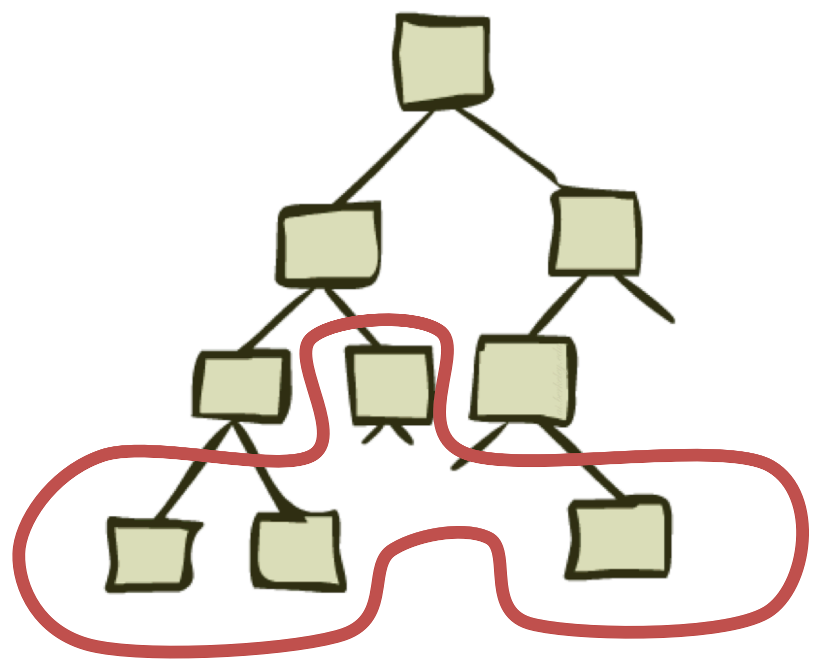
How big is its search tree (from S)?

$$\infty$$

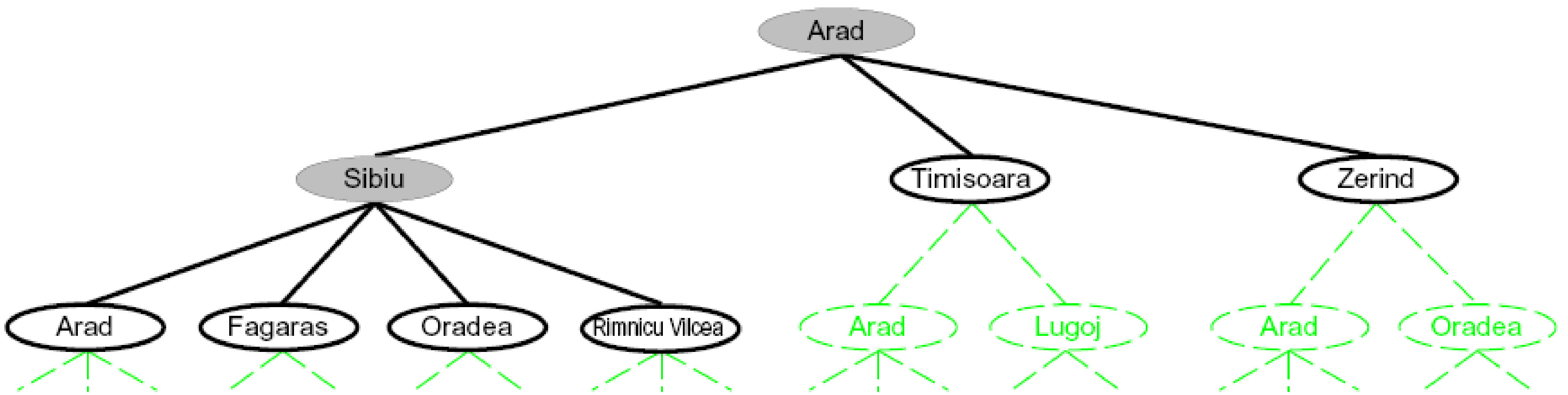
Searching for Solutions

Basic idea: incrementally build a search tree until a goal state is found

- Root = initial state
- Expand via transition function to create new nodes
- Nodes that haven't been expanded are **leaf nodes** and form the **frontier** (**open list**)
- Different **search strategies** (next lecture) choose next node to expand (as few as possible!)
- Use a **closed list** to prevent expanding the same state more than once

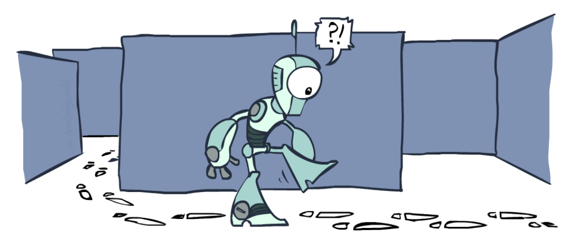


General Algorithm




```
function GRAPH-SEARCH(problem, fringe) returns a solution, or failure
   closed ← an empty set
   fringe ← INSERT(MAKE-NODE(INITIAL-STATE[problem]), fringe)
   loop do
      if fringe is empty then return failure
      node ← REMOVE-FRONT(fringe)
      if GOAL-TEST(problem, STATE[node]) then return node
      if STATE[node] is not in closed then
         add STATE[node] to closed
         fringe ← INSERTALL(EXPAND(node, problem), fringe)
   end
```

Queue (FIFO)
Stack (LIFO)
Priority Queue



Evaluating a Search Strategy

Solution

- **Completeness**: does it always find *a* solution if one exists?
- **Optimality**: does it always find a least-cost solution?

Efficiency

- **Time Complexity**: number of nodes generated/expanded
- **Space Complexity**: maximum number of nodes in memory

Computational Complexity (A.1)

- We are going to be comparing several algorithms
  - How do we tell if one is faster/leaner than another?

- **Benchmarking** involves running the algorithm on a computer and measuring performance (e.g. time in sec, memory in bytes)
  - Unsatisfactory: specific to machine, implementation, compiler, inputs, …

- **Complexity Analysis** is a mathematical approach that abstracts away from these details

Asymptotic Analysis

Basic idea: get a sense of "rate of growth" of an algorithm, which tells us how "bad" it will get as problem size grows

Example

```
def summation(l):
    sum = 0
    for n in l:
        sum += n
    return sum
```

```
>>> summation(range(5))
10
```

Step 1: Identify Size Parameter

- We need to abstract over the input and just identify what parameter characterizes the size of the input

```
def summation(l):
    sum = 0
    for n in l:
        sum += n
    return sum
```

- For the example what matters is the length of the input list
  - We'll refer to this as *n*

Step 2: Identify Performance Measure

- Again, abstract over the implementation and find a measure that reflects running time (or memory usage), not tied to a particular computer

```
def summation(l):
    sum = 0
    for n in l:
        sum += n
    return sum
```

- In this case it could be lines executed, or operations (additions, assignments) performed
  - Call this f(n)

If f(n) measures lines executed?

f(n) = 2n + 2

Step 3: Identify Comparison Metric

- It is typically not possible to identify *exactly* the size parameter (i.e. one that perfectly characterizes the performance), and so we settle for a representative metric

- Most common is **worst case**
  – Sometimes best case, average case

Step 4: Approximation

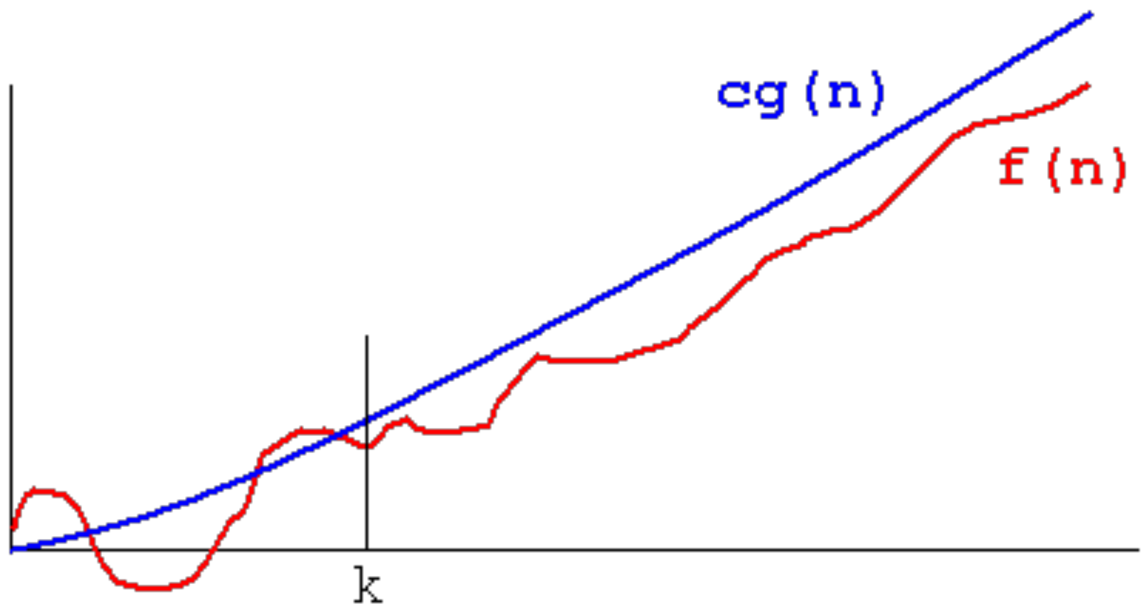
- Typically it is hard to *exactly* compute f(*n*), and so we settle for an approximation
- For worst-case, **Big-O notation**, O(), yields this formal asymptotic analysis…

$$f(n) = \mathcal{O}(g(n)) \text{ as } n \to \infty$$

$$\equiv \exists\, c \in \mathbb{N}, k \in \mathbb{N} \ \text{ s.t.}$$

$$\forall n > k \quad |f(n)| \leq c|g(n)|$$

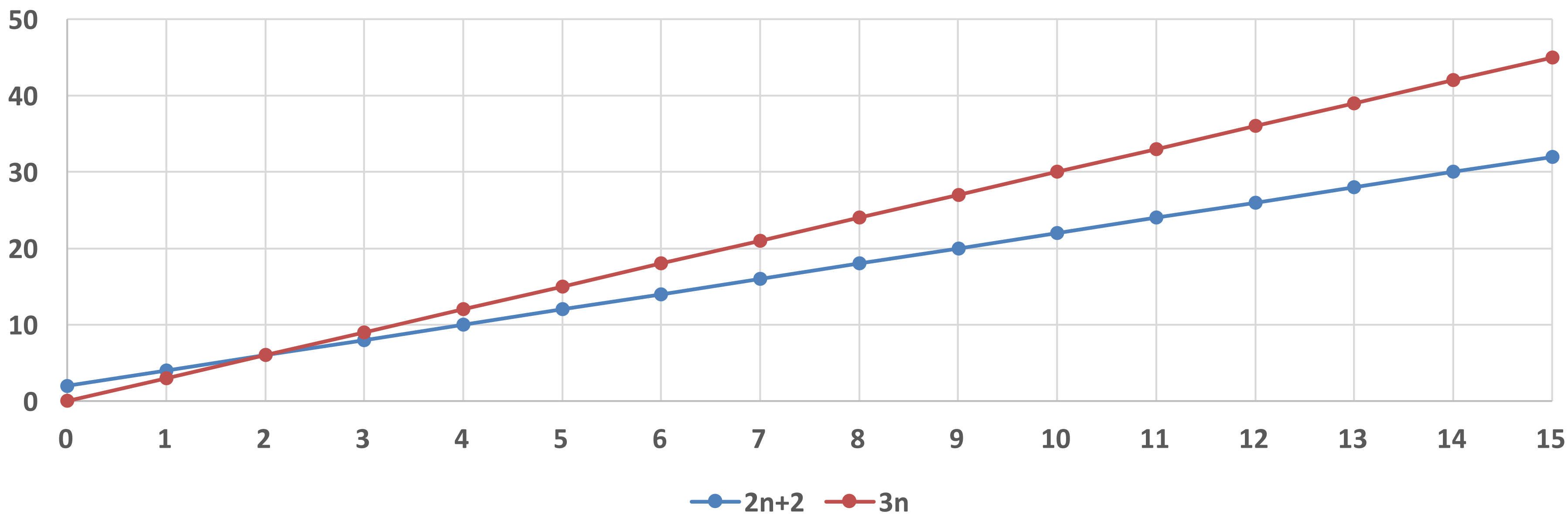
Big-O Definition Visually



Example

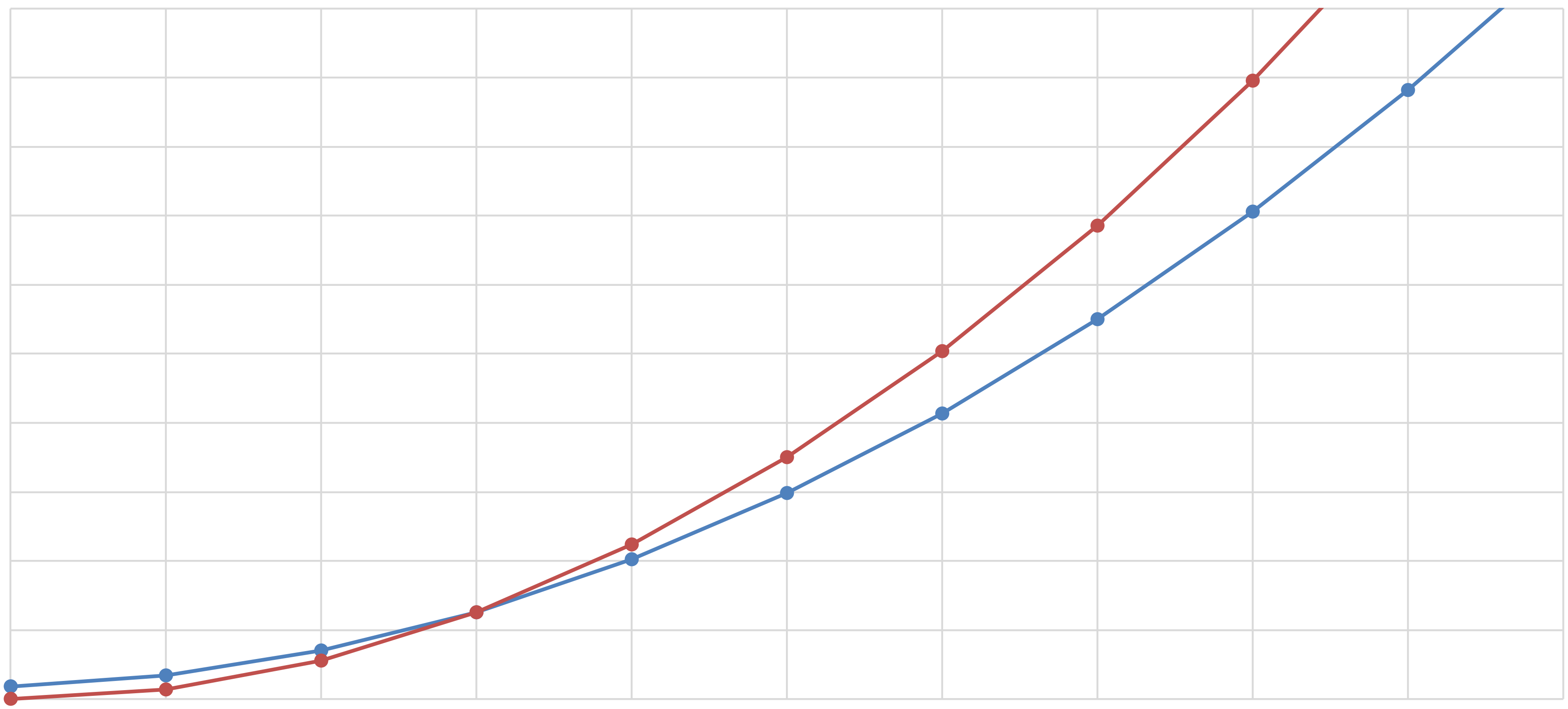
- Since f(n) = 2n + 2, we can show that this function is O(n)
  - c=3, k=2

```
def summation(l):
    sum = 0
    for n in l:
        sum += n
    return sum
```



Exercise

Prove: $5n^2 + 3n + 9 = O(n^2)$



Solution

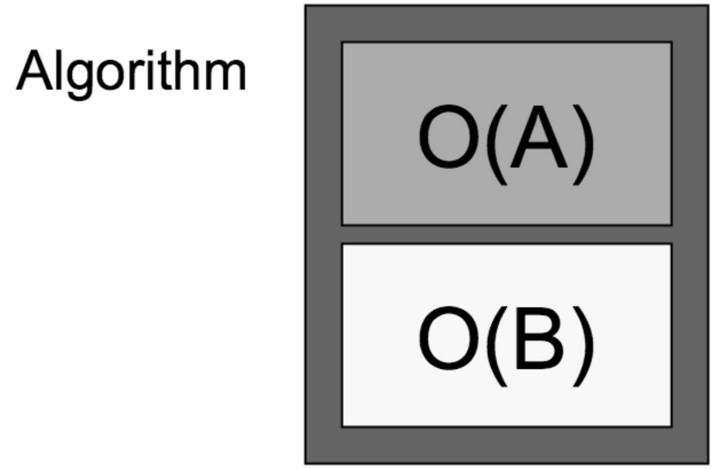
Find c and k such that…

$$\forall n > k \quad cn^2 > 5n^2 + 3n + 9$$

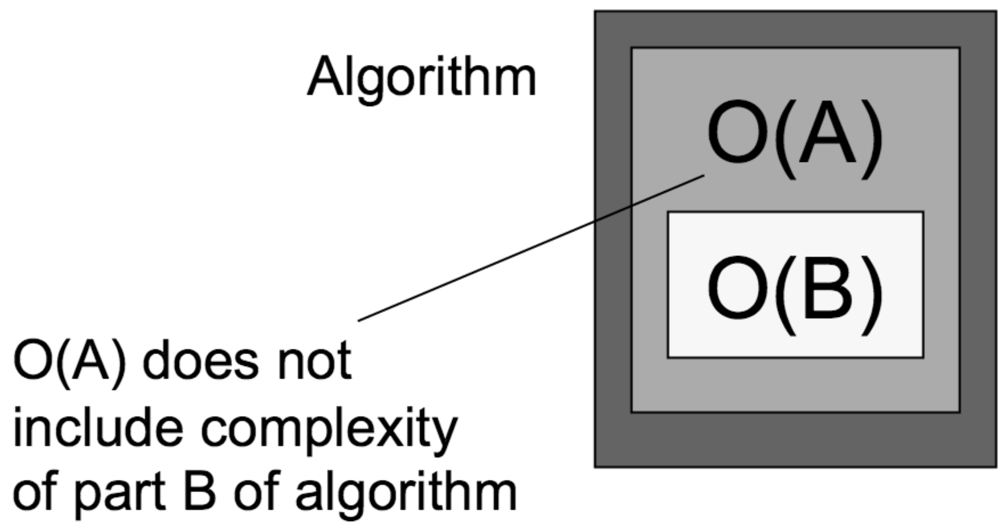
1. Solve: $cn^2 = 5n^2 + 3n + 9$
2. Let n=k, solve: $c = 5 + \frac{3}{k} + \frac{9}{k^2}$
   – If k=3, c=7
3. So… $7n^2 > 5n^2 + 3n + 9 \quad \forall n > 3$
   – And thus… $5n^2 + 3n + 9 = \mathcal{O}(n^2)$

Order of Complexity

- O(A) + O(B) = max(O(A), O(B))
  - Slower parts of an algorithm dominate faster parts

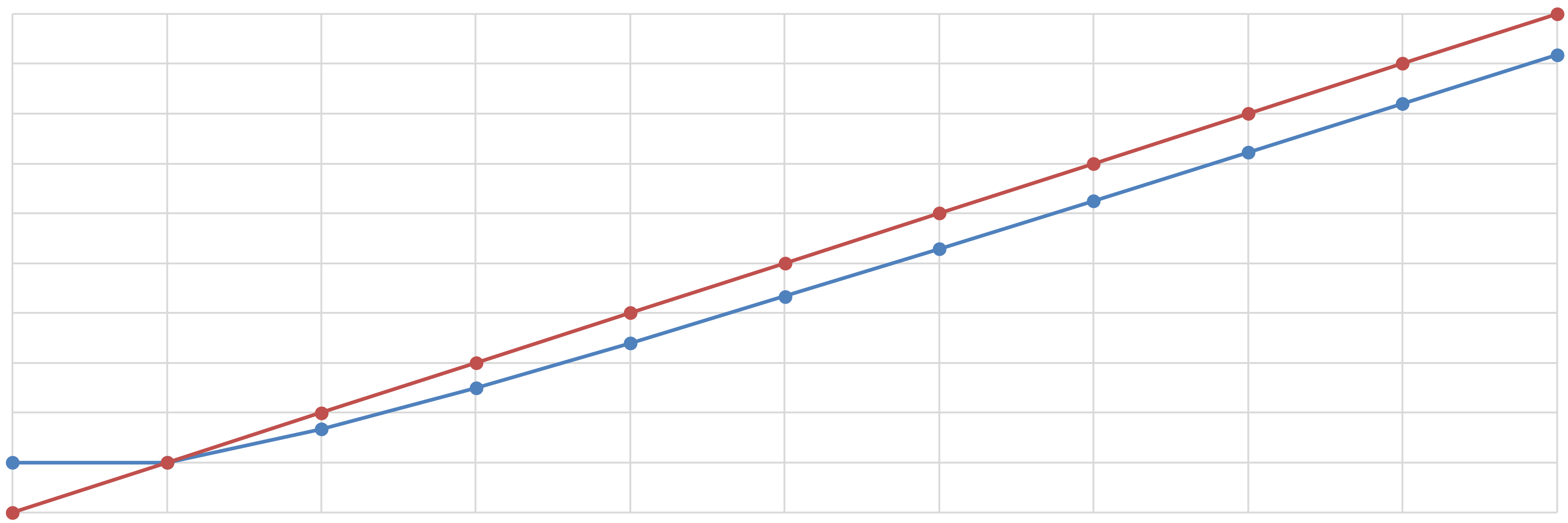


- O(A) * O(B) = O(A*B)
  - Nesting



Exercise

Prove: $$\frac{x^2+1}{x+1} = \mathcal{O}(x)$$



Solution

$$\mathcal{O}(\frac{x^2+1}{x+1}) = \frac{\mathcal{O}(x^2+1)}{\mathcal{O}(x+1)}$$

$$= \frac{\mathcal{O}(x^2)}{\mathcal{O}(x)}$$

$$= \mathcal{O}(\frac{x^2}{x})$$

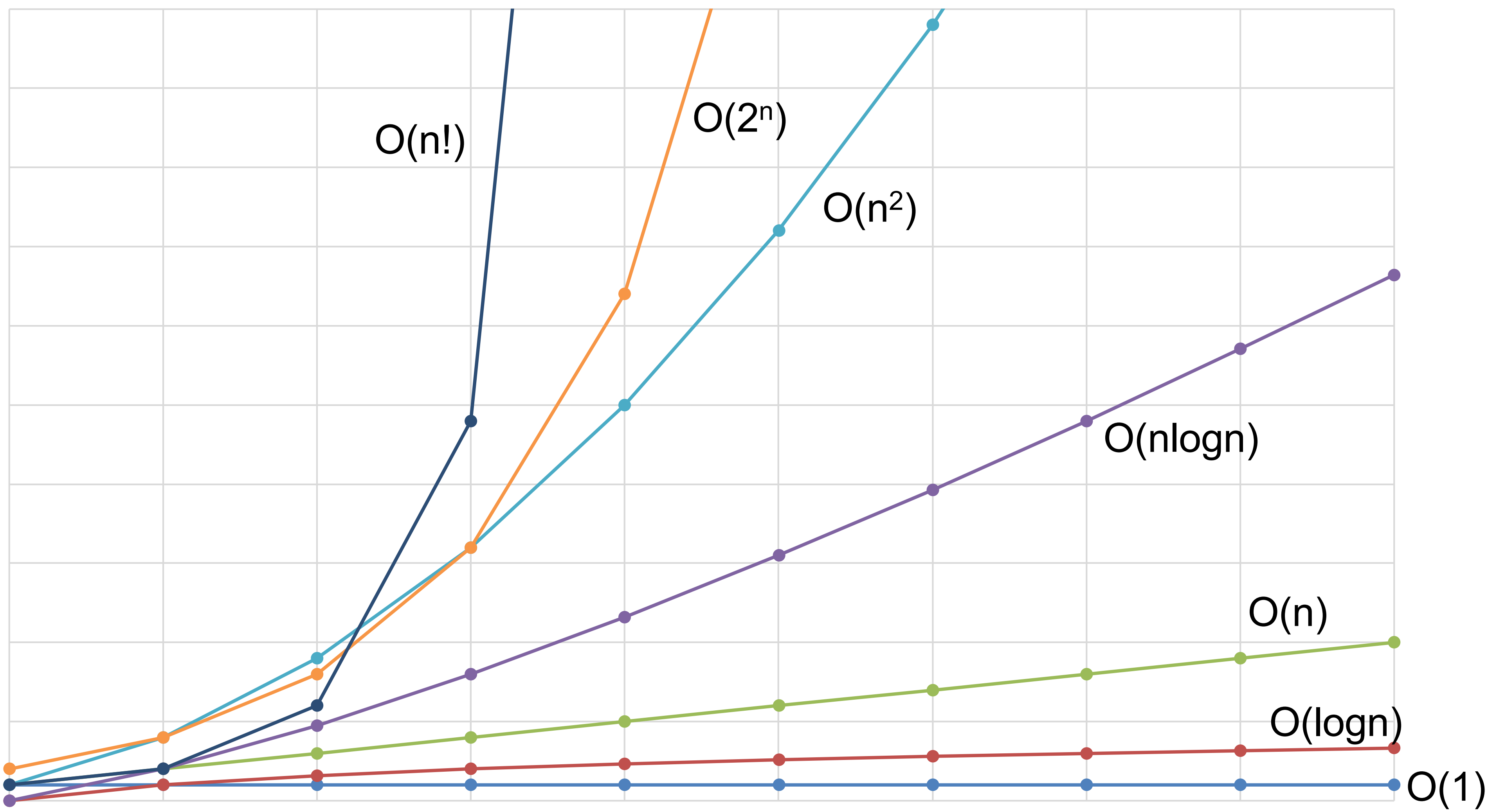
$$= \mathcal{O}(x)$$

Big-O Numerically

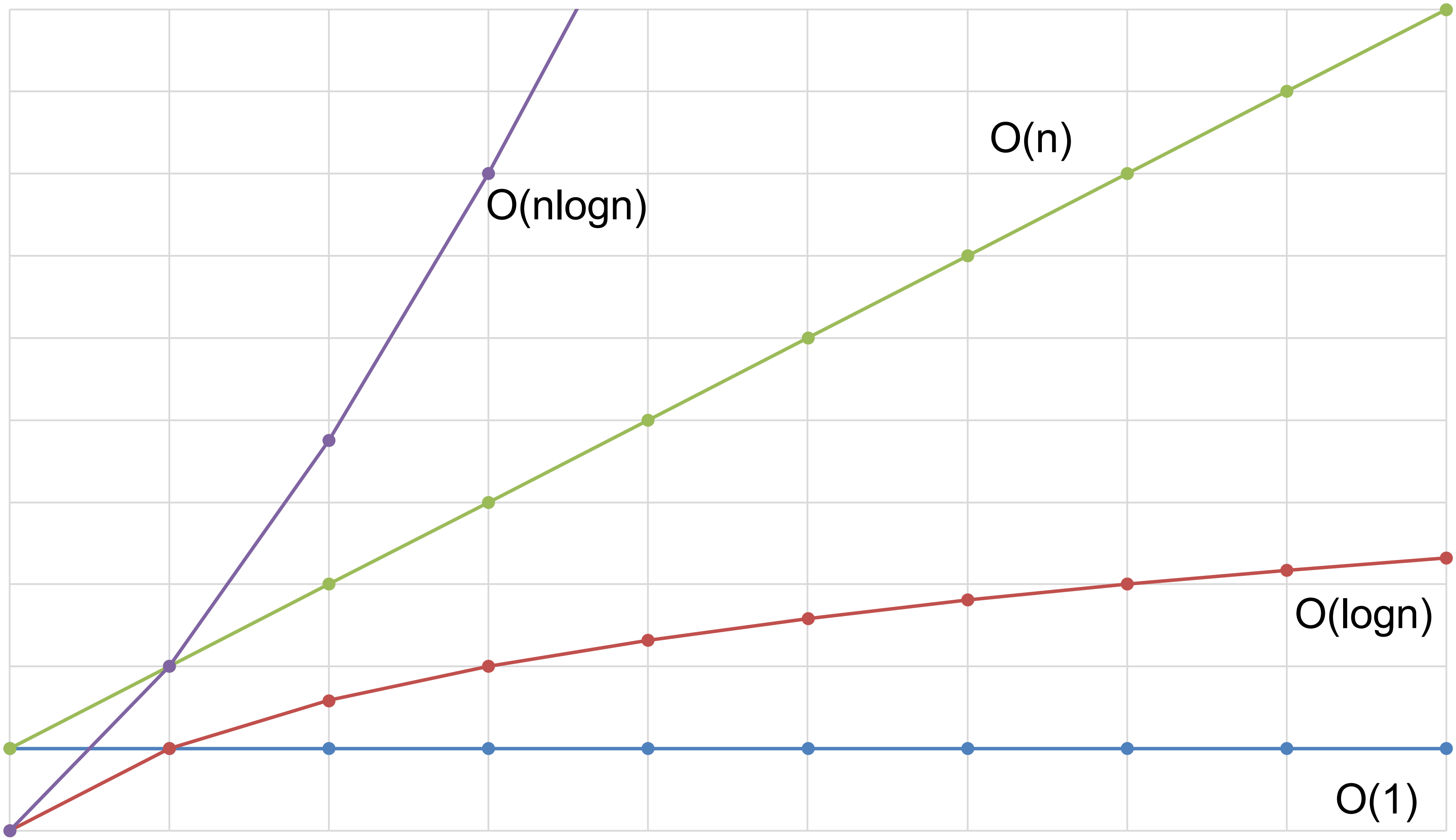
Big-O	Term	Cost for n=10
O(1)	Constant	1
O(log n)	Logarithmic	3
O(n)	Linear	10
O(n log n)	Log-Linear, Linearithmic	33
$O(n^2)$	Quadratic	100
$O(2^n)$	Exponential	1,024
O(n!)	Factorial	3,628,800

It's important to know this ranking of growth!

Asymptotic Visual



Asymptotic Visual (zoom)



Exercise

Choose **all** g(n) for which the following statement is true:

$$10n^2 + 14n - 12 = O(g(n))$$

- 1
- log(n)
- n
- nlogn
- n^2
- 2^n
- n!

Answer

Choose **all** g(x) for which the following statement is true:

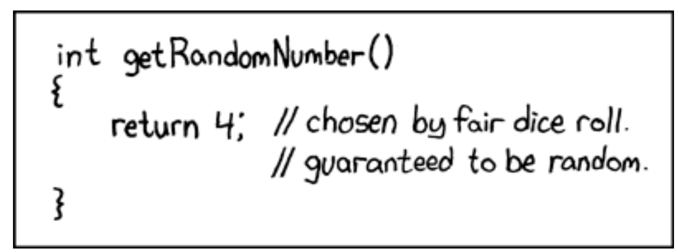
$$10n^2 + 14n - 12 = O(g(n))$$

- 1
- log(n)
- n
- nlogn
- **n^2**
- **2^n**
- **n!**

Example: O(1)

Stays constant regardless of problem size

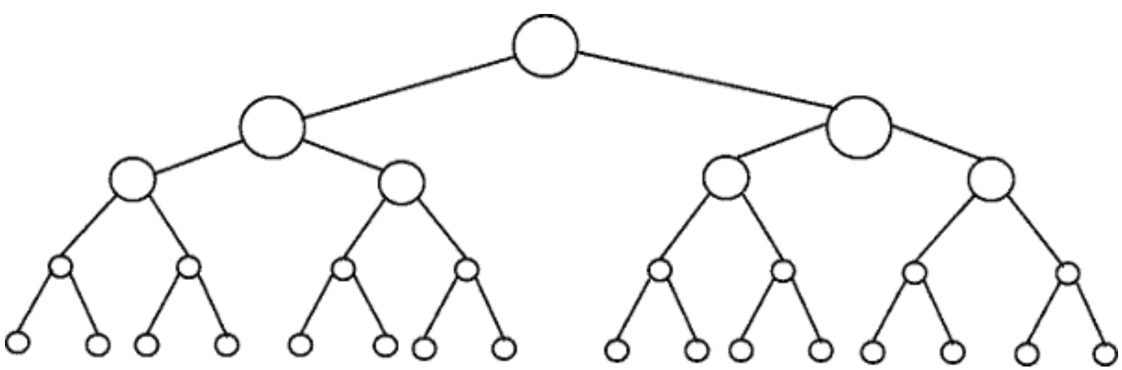
- Check even/odd
- Hash computation
- Array indexing

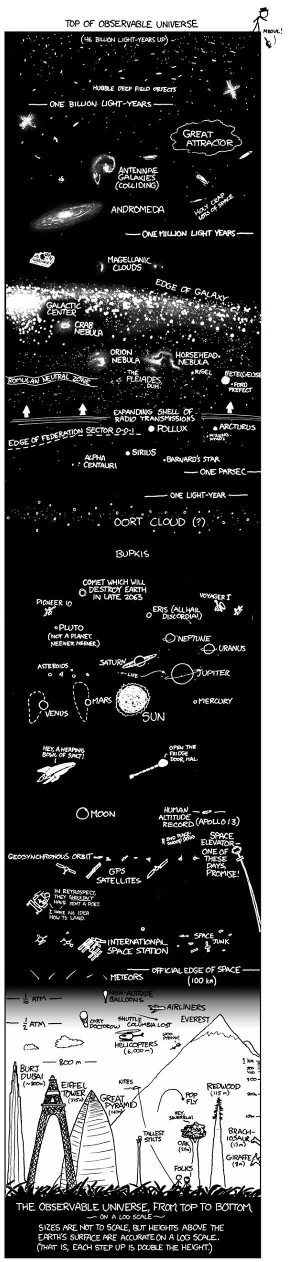


Example: O(logn)

Inverse of exponential: as you double the problem size, resource consumption increases by a constant

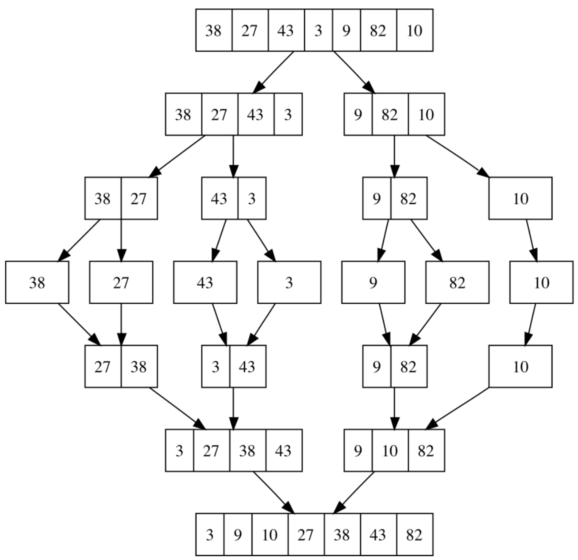
- Binary search
- Balanced tree search

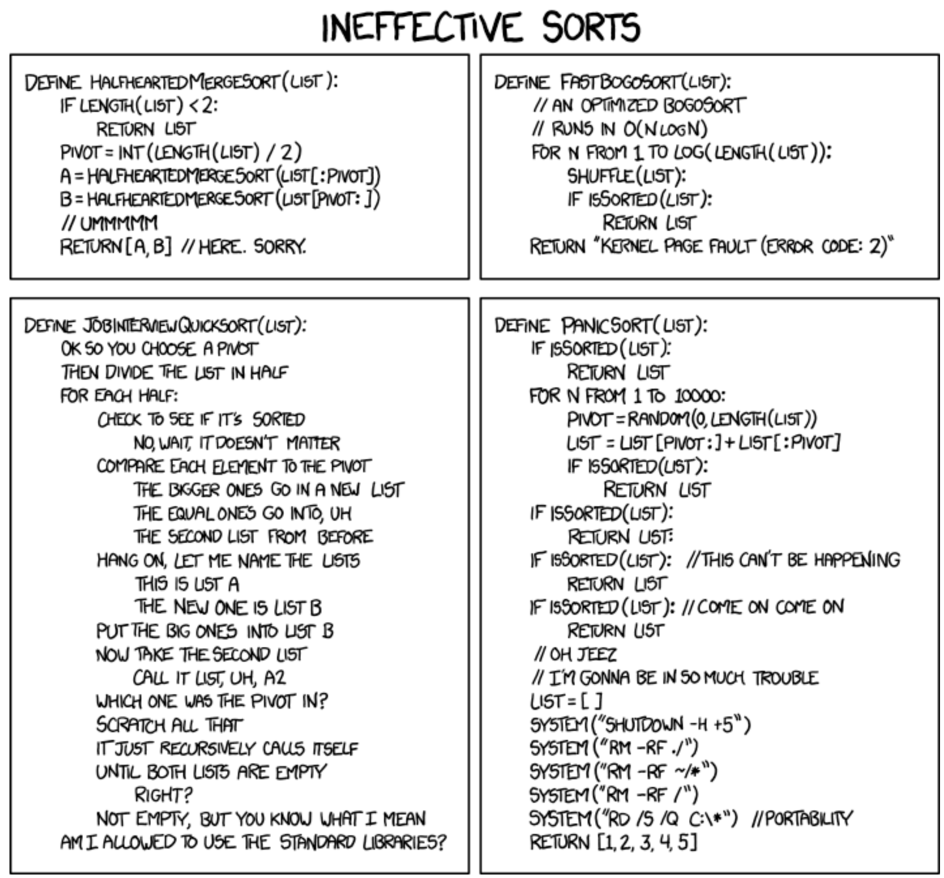




Example: O(n log n)

- Performing an O(log n) operation for each item in your input
  - Typical of efficient sorting





Example: $O(n^2)$

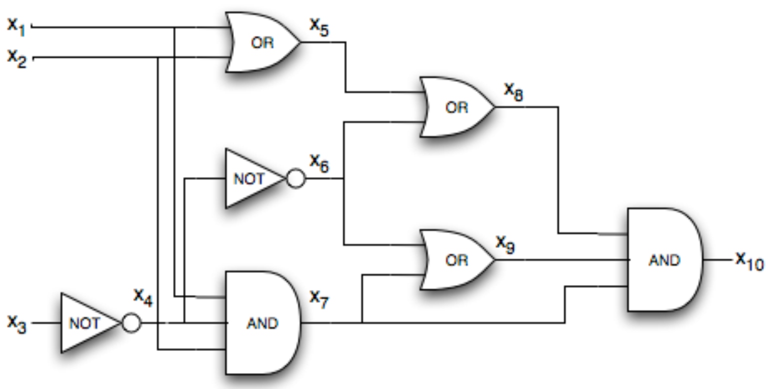
For each item, perform an operation with each other item

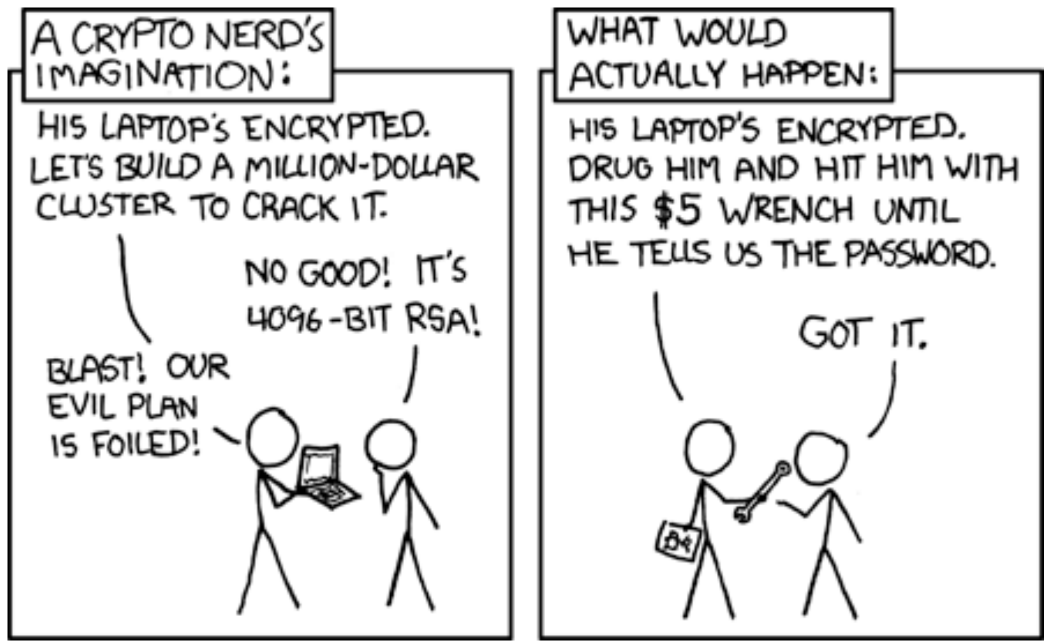
- Duplication detection
- Pairwise comparison
- Bubble sort

Example: $O(2^n)$

For every added element, resource consumption doubles

- Hardware verification
- Cryptography

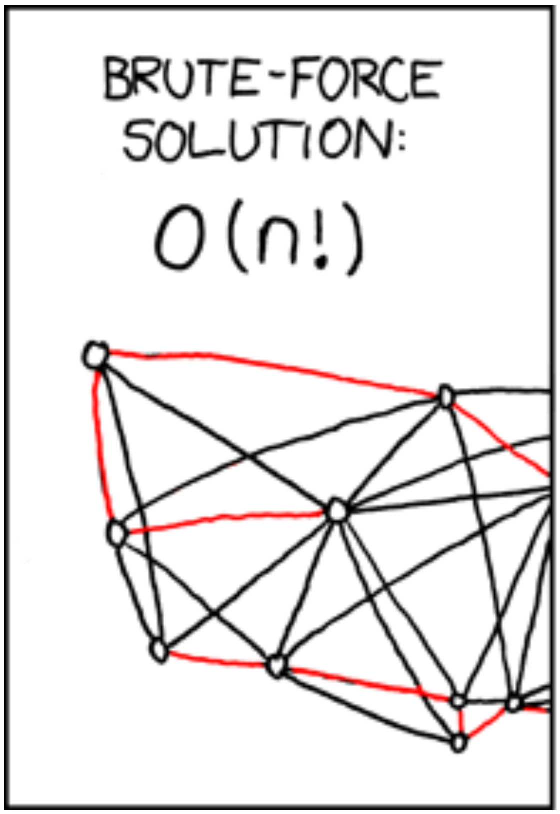




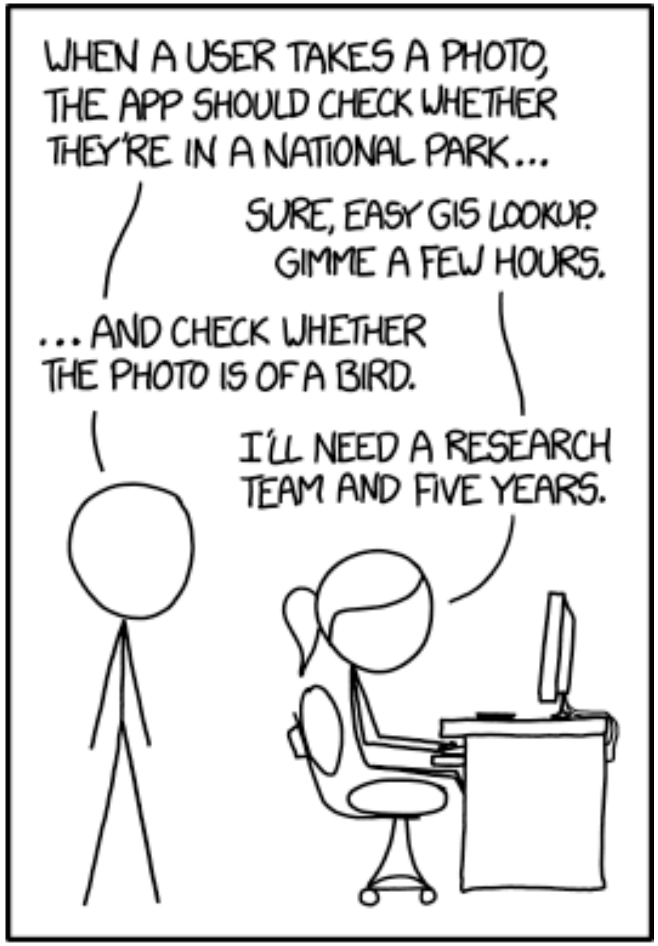
Complexity Analysis

- Thus far we have analyzed algorithms, but **complexity analysis** focuses on problems, and classes of problems
- Problems that can be solved in polynomial time, $O(n^k)$, form class **P**
  - Generally considered "easy" (but could have large *c*)
- Problems in which you can verify a solution in polynomial time form **NP**
  - The "hardest" in NP are **NP-complete**
- Open question: $\mathrm{P} \stackrel{?}{=} \mathrm{NP}$
  - Most computer scientists assume not
  - If correct, there can be no algorithm that solves *all* such problems in polynomial time
  - AI is interested in developing algorithms that perform efficiently on *typical* problems drawn from a pre-determined distribution

Complex[ity] Humor (TSP)



Complex[ity] Humor (AI)



IN CS, IT CAN BE HARD TO EXPLAIN THE DIFFERENCE BETWEEN THE EASY AND THE VIRTUALLY IMPOSSIBLE.

Summary

- We can represent deterministic, fully observable, discrete, known tasks as **search problems**
  - **Initial state**, **transition function**, **goal test**, **path cost**
    - **State space**: all states reachable from initial
  - **Solution**: action sequence, initial->goal
    - **Optimal**: least path cost
- We **abstract** search state representation depending on the search problem for computational tractability
- Once formulated, we solve a search problem by incrementally forming a **search tree** until a goal state is found
  - We evaluate algorithms with respect to solution **completeness/optimality** and time/space **complexity**
  - More next lecture!